An obstruction for q-deformation of the convolution product

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Abstract. We consider two independent q-Gaussian random variables X_0 and X_1 and a function γ chosen in such a way that $\gamma(X_0)$ and X_0 have the same distribution. For $q \in (0, 1)$ we find that at least the fourth moments of $X_0 + X_1$ and $\gamma(X_0) + X_1$ are different. We conclude that no q-deformed convolution product, parallelling the known cases $q = 0$ and $q = 1$, can exist.

1. Introduction and notation

In 1982 Voiculescu discovered a new notion of statistical independence, which he called 'freeness'. Two self-adjoint operators X_1 and X_2 on a Hilbert space $\mathcal{H}$ are said to be free, or freely independent, with respect to a state vector $\xi \in \mathcal{H}$ if for all $n \in \mathbb{N}$ and all bounded and measurable functions $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ such that if $\langle \xi, f_j(X_{i_j})\xi \rangle = 0$ for all j one has:

$$\langle \xi, f_1(X_{i_1}) \cdots f_n(X_{i_n})\xi \rangle = 0$$

where $i_1, \ldots, i_n$ is an alternating sequence of 1s and 2s, i.e. $i_1 \neq i_2 \neq \cdots \neq i_n$. It turned out that this notion of independence brought along with it its own convolution product and its own stable laws. In particular, the unique freely stable law of finite variance is Wigner's semicircle law, the free analogue of the Gaussian distribution. Between these two cases, classical independence and free independence, there exists a natural interpolation indexed by a parameter $q \in [-1, 1]$, where $q = 0$ corresponds to freeness and $q = 1$ to classical independence. The connection is formed by the q-harmonic oscillator. Namely, on the one hand the Gauss measure coincides with the ground state probability distribution of the quantum harmonic oscillator. On the other hand, the ground state probability distribution of $S + S^*$, where S is the left shift on $l^2(\mathbb{N})$, is precisely Wigner's semicircle law. The operator S on $l^2(\mathbb{N})$ can be viewed as the annihilation operator of a q-harmonic oscillator for $q = 0$. It is therefore natural to investigate the general q-harmonic oscillator and the associated quantum field, as a candidate for a new, intermediate form of independence and white noise in quantum mechanics. In [3] it was shown that this q-quantum field has the functorial character of a second quantization. In particular, it follows that its ground state probability distribution, ν_q, is a stable law for addition of quantum fields: let X_1 be a field variable with distribution ν_q, dilated to variance σ_1^2 and let X_2 be a field variable with distribution ν_q dilated to variance σ_2^2 such that X_1 and X_2 are q-independent. Then $X_1 + X_2$ is ν_q-distributed with variance $\sigma_1^2 + \sigma_2^2$. For this reason ν_q is called the q-Gaussian distribution (see [3, 9]).

It seems important to further investigate what properties the intermediate q-cases share with the standard cases $q = 0$ and $q = 1$. In particular: is there a good q-convolution?

Suppose X_1 and X_2 are independent random variables in the classical sense. Then the distribution μ of their sum is determined by the distributions μ_1 of X_1 and μ_2 of X_2. Indeed, μ is the convolution product of μ_1 and μ_2. If, in the above case, we replace 'independent' by 'free' (or 'freely independent'), then again μ is determined by μ_1 and μ_2. This defines the free convolution product $\mu_1 \boxplus \mu_2 := \mu$ of μ_1 and μ_2. Free convolution is an interesting operation involving Cauchy transforms and inverted functions [10, 12].

Let us now consider the same situation for general q. First we must specify what we mean by 'q-independent' random variables X_1 and X_2, other than the known q-Gaussian ones. It seems reasonable to call functions of X_1 and X_2 q-independent if X_1 and X_2 are q-independent q-Gaussians. In this paper we show that this runs into the following difficulty: the distribution of the sum of such q-independent random variables is no longer determined by the distributions of the summands. This will be shown by means of a counter-example.

In 1991 Bożejko and Speicher introduced the q-quantum field (cf [1, 2]). Their construction is based on a q-deformation, $\mathcal{F}_q(\mathcal{H})$, of the full Fock space over a separable Hilbert space $\mathcal{H}$. Their random variables are given by self-adjoint operators of the form

$$X(f) := a(f) + a(f)^* \qquad f \in \mathcal{H}$$

where $a(f)$ and $a(f)^*$ are the annihilation and creation operators associated with f satisfying the q-deformed commutation relation

$$a(f)a(g)^* - qa(g)^*a(f) = \langle f, g \rangle \mathbb{1}. \tag{1}$$

This commutation relation was first introduced by Frisch and Bourret in [5] and various aspects of it were studied in [4, 6, 7, 9].

For $q = 1$ the random variables $X(f)$ and $X(g)$, with $f \perp g$, are independent Gaussian random variables in the classical sense, in the limit $q \downarrow 0$ they become freely independent in the sense of Voiculescu [12].

The construction of the Fock representation for (1) is described in [1, 4], but for completeness we give the necessary details here. Operators $a(f)$ and $a(f)^*$ are, for all $f \in \mathcal{H}$, defined on the full Fock space $\mathcal{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\otimes n}$ by:

$$a(f)^* h_1 \otimes \cdots \otimes h_n := f \otimes h_1 \otimes \cdots \otimes h_n \qquad n \in \mathbb{N}, h_1, \ldots, h_n \in \mathcal{H}$$

and

$$a(f)\Omega := 0$$

$$a(f)h_1 \otimes \cdots \otimes h_n := \sum_{k=1}^{n} q^{k-1} \langle f, h_k \rangle h_1 \otimes \cdots \check{h}_k \cdots \otimes h_n \qquad n \geqslant 1 \tag{2}$$

where the notation $h_1 \otimes \cdots \check{h}_k \cdots \otimes h_n$ stands for the tensor product $h_1 \otimes \cdots \otimes h_{k-1} \otimes h_{k+1} \otimes \cdots \otimes h_n$ and $\Omega = 1 \oplus 0 \oplus 0 \oplus \cdots$. In order to ensure that $a(f)^*$ is the adjoint of $a(f)$ for all $f \in \mathcal{H}$, Bożejko and Speicher recursively define an inner product $\langle \cdot, \cdot \rangle_q$ on $\mathcal{F}$ as

$$\langle g_1 \otimes \cdots \otimes g_m, h_1 \otimes \cdots \otimes h_n \rangle_q = \delta_{n,m} \langle g_2 \otimes \cdots \otimes g_m, a(g_1)h_1 \otimes \cdots \otimes h_n \rangle_q$$

$$= \delta_{n,m} \sum_{k=1}^{n} q^{k-1} \langle g_1, h_k \rangle \langle g_2 \otimes \cdots \otimes g_m, h_1 \otimes \cdots \check{h}_k \cdots \otimes h_n \rangle_q.$$

We denote the full Fock space $\mathcal{F}$ equipped with this inner product by $\mathcal{F}_q(\mathcal{H})$. By a well known theorem of Gelfand, Naimark and Segal, known as the GNS construction, there exists, up to unitary equivalence, only one cyclic representation of the relations (1)

and (2). For $\mathcal{H} = \mathbb{C}$ the above construction reduces to $\mathcal{F}_q(\mathbb{C}) \cong l^2(\mathbb{N}, [n]_q!)$, where $[n]_q = (1 - q^n)/(1 - q)$ and $[n]_q! = \prod_{j=1}^{n}[j]_q$ with $[0]_q! = 1$.

In [2,9] the density of the q-Gaussian distribution, $\nu_q(\mathrm{d}x)$, of the random variable $X_0 = a(f_0) + a(f_0)^*$ with $f_0 \in \mathcal{H}$ and $\|f_0\| = 1$ is calculated. This density is a measure on $\mathbb{R}$, where it is supported on the interval $[-2/\sqrt{1 - q}, 2/\sqrt{1 - q}]$. If we denote the n-fold product $\prod_{k=0}^{n-1}(1 - aq^k)$ by $(a; q)_n$ and agree on $(a_1, \ldots, a_m; q)_n = (a_1; q)_n \cdots (a_m; q)_n$, then $\nu_q(\mathrm{d}x)$ can be written as

$$\nu_q(\mathrm{d}x) = \nu_q'(x)\,\mathrm{d}x = \frac{1}{\pi}\sqrt{1 - q}\,\sin\theta\,(q, qv^2, qv^{-2}; q)_\infty\,\mathrm{d}x$$

where $2\cos\theta = x\sqrt{1 - q}$ and $v = \exp(i\theta)$.

To state the main theorem of this paper we define X_1 to be the random variable $a(f_1) + a(f_1)^*$ for some $f_1 \in \mathcal{H}$ with $\|f_1\| = 1$ and $\langle f_0, f_1 \rangle = 0$. Then X_0 and X_1 are q-Gaussian random variables, independent in the sense of quantum probability:

$$\langle \Psi, \beta_0(X_0)\beta_1(X_1)\Psi \rangle_q = \langle \Psi, \beta_1(X_1)\beta_0(X_0)\Psi \rangle_q = \langle \Psi, \beta_0(X_0)\Psi \rangle_q \langle \Psi, \beta_1(X_1)\Psi \rangle_q$$

for bounded and measurable functions $\beta_0, \beta_1 : \mathbb{R} \to \mathbb{R}$. See [1,8].

Theorem 1. There exists a function $\gamma : \mathbb{R} \to \mathbb{R}$ such that X_0 and $\gamma(X_0)$ are identically distributed but $X_0 + X_1$ and $\gamma(X_0) + X_1$ are not.

The consequence of this theorem is that the distribution of the sum of two or more random variables depends on the choice of random variables and not solely on the respective distributions of these random variables. This means that a q-convolution parallelling the known convolution for probability measures for the cases $q = 0$ (cf [10, 12]) and $q = 1$ cannot exist.

In contrast to the above, Nica [11], constructs a convolution law for probability distributions that interpolates between the known cases $q = 0$ and $q = 1$. Theorem 1 implies that this interpolation does not hold for functions of q-Gaussians. In fact this can also be seen by explicit calculation of the moments of the distribution of $X_0^n + X_1^m$, $n, m \in \mathbb{N}$, using the convolution law Nica suggests and using the structure inherently present in $\mathcal{F}_q(\mathcal{H})$. From the fourth moment onwards the moments differ for $n, m > 1$, although they are the same for the cases $q = 0$ and $q = \pm 1$, as they should be.

In the next section we shall prove theorem 1 by constructing the function γ and showing that the fourth moment of the distribution of $\gamma(X_0) + X_1$ is strictly smaller then the fourth moment of the distribution of $X_0 + X_1$ for $q \in (0, 1)$.

2. Construction of γ and proof of theorem

In [9] we construct the unitary operator $U : \mathcal{F}_q(\mathbb{C}) \to L^2(\mathbb{R}, \nu_q)$ that diagonalizes the operator $X = a + a^*$ with $a = a(1)$, such that $UX = TU$ with T the operator of pointwise multiplication on $L^2(\mathbb{R}, \nu_q)$ given by $(Tf)(x) = xf(x)$ for $f \in L^2(\mathbb{R}, \nu_q)$.

Let γ be the transformation on $[-2/\sqrt{1 - q}, 2/\sqrt{1 - q}]$ that changes the orientations of $[-2/\sqrt{1 - q}, 0]$ and $[0, 2/\sqrt{1 - q}]$ in such a way that the distribution ν_q is preserved. For this γ has to satisfy the differential equation

$$\nu_q'(x)\,\mathrm{d}x + \nu_q'(\gamma(x))\,\mathrm{d}\gamma(x) = 0 \tag{3}$$

with $\gamma(-2/\sqrt{1 - q}) = \gamma(2/\sqrt{1 - q}) = 0$. Indeed, this leads to

$$\mathbb{P}(0 \leqslant T \leqslant x) = \mathbb{P}(0 \leqslant \gamma(T) \leqslant x) = \mathbb{P}(\gamma^{-1}(x) \leqslant T \leqslant 2/\sqrt{1 - q})$$
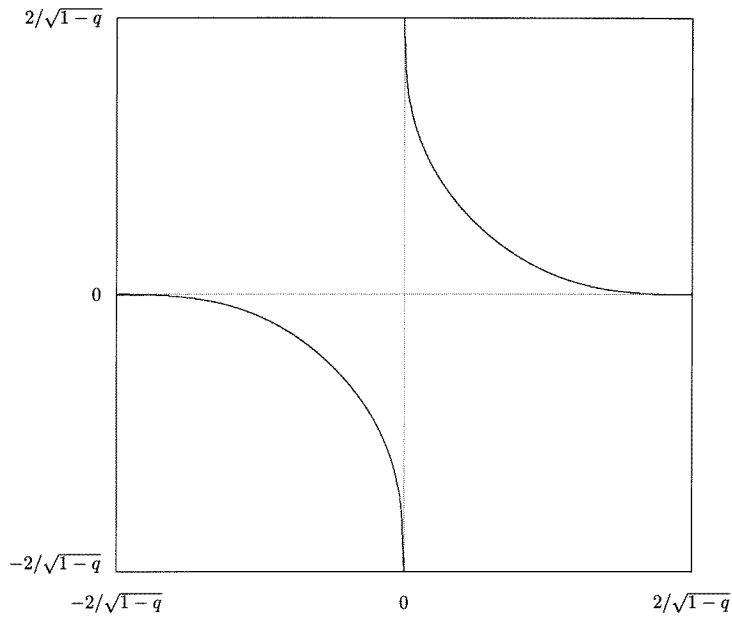
Figure 1. The function γ.

as can be seen by differentiating both sides with respect to x. Note that the function γ is its own inverse. Figure 1 shows a typical picture of the shape of the function γ.

Let $\hat{W}$ be the unitary operator on $L^2(\mathbb{R}, \nu_q)$ that implements γ:

$$(\hat{W}f)(x) = f(\gamma(x)) \qquad \text{for } f \in L^2(\mathbb{R}, \nu_q).$$

This immediately implies that $\hat{W}^2 = \mathbb{1}$ since $\gamma \circ \gamma = \text{id}$ so $\hat{W}$ is self-adjoint. If we define $\tilde{W} := U^*\hat{W}U$ it follows that

$$\gamma(X) = \gamma(U^*TU) = U^*\gamma(T)U = U^*\hat{W}T\hat{W}U = \tilde{W}X\tilde{W}$$

so $\tilde{W}$ is a unitary and self-adjoint operator on $\mathcal{F}_q(\mathbb{C})$ that implements γ on X. Note that $\tilde{W}\Omega = \Omega$ because $\tilde{W}1 = 1$. On the canonical basis $(e_j)_{j \in \mathbb{N}}$ of $\mathcal{F}_q(\mathbb{C})$ the operator $\tilde{W}$ can be written as

$$\tilde{W}e_n = \sum_{k=0}^{\infty} w_{kn}e_k \qquad \text{with } w_{00} = 1.$$

Now let us choose $\mathcal{H} = \mathbb{C}^2$, $f_0 = (1, 0)$ and $f_1 = (0, 1)$ and let us denote $a(f_0)$ by a_0 and $a(f_1)$ by a_1. Recall from the introduction that $X_0 = a_0 + a_0^*$ and $X_1 = a_1 + a_1^*$. In this setting we need a unitary operator W on $\mathcal{F}_q(\mathbb{C}^2)$ that satisfies $\gamma(X) = WXW$. To this end we denote by $\mathcal{K} \subset \mathcal{F}_q(\mathbb{C}^2)$ the kernel of the operator a_0 on $\mathcal{F}_q(\mathbb{C}^2)$. Then by constructing an isomorphism $V: \mathcal{F}_q(\mathbb{C}) \otimes \mathcal{K} \to \mathcal{F}_q(\mathbb{C}^2)$, the operator $\tilde{W}$ can be extended to $W = V(\tilde{W} \otimes \mathbb{1})V^*$.

Proposition 1. The space $\mathcal{F}_q(\mathbb{C}^2)$ is canonically isomorphic to $\mathcal{F}_q(\mathbb{C}) \otimes \mathcal{K}$.

Proof. From the commutation relation (1) we find that $a_0^n(a_0^*)^n = P_n(a_0^*a_0)$, where P_n is a polynomial of degree n with constant coefficient $[n]_q!$. In fact, P_n is given by

$$P_n(x) = \prod_{j=1}^{n}(q^j x + [j]_q).$$

For $n \in \mathbb{N}$, let $\mathcal{K}_n$ denote the Hilbert subspace $(a_0^*)^n \mathcal{K}$. Note that $\mathcal{K}_n$ is indeed closed, since $(a_0^*)^n$ acts on $\mathcal{K}$ as a multiple of an isometry, for every $\varphi \in \mathcal{K}$,

$$\|(a_0^*)^n \varphi\|_q^2 = \langle \varphi, P_n(a_0^* a_0)\varphi \rangle_q = [n]_q! \|\varphi\|_q^2.$$

Furthermore, $\mathcal{K}_n \perp \mathcal{K}_m$ for $n > m$, since for $\varphi, \psi \in \mathcal{K}$,

$$\begin{aligned}
\langle (a_0^*)^n \varphi, (a_0^*)^m \psi \rangle_q &= \langle (a_0^*)^{n-m} \varphi, a_0^m (a_0^*)^m \psi \rangle_q \\
&= \langle (a_0^*)^{n-m} \varphi, P_m(a_0^* a_0)\psi \rangle_q \\
&= [m]_q! \langle \varphi, (a_0)^{n-m} \psi \rangle_q = 0.
\end{aligned}$$

Now suppose that some $\psi \in \mathcal{F}_q(\mathbb{C}^2)$ is orthogonal to all the $\mathcal{K}_n$. We claim that for all $n \in \mathbb{N}$

$$\psi \perp \ker a_0^n \tag{4}$$

from which it follows that $\psi = 0$, since $(\mathbb{C}^2)^{\otimes n} = \mathcal{F}_q^{(n)}(\mathbb{C}^2) \subset \ker a_0^{n+1}$.

We proceed to prove (4) by induction. For $n = 1$ we already have (4) since $\ker a_0 = \mathcal{K}$. Suppose that (4) holds for some n. Then $\psi \in \overline{\text{Ran}(a_0^*)^n}$, say $\psi = \lim_{k \to \infty} (a_0^*)^n \varphi_k$ with $\varphi_k \in \mathcal{F}_q(\mathbb{C}^2)$, $k \in \mathbb{N}$. Take $\theta \in \ker a_0^{n+1}$ and define $\xi := a_0^n \theta \in \mathcal{K}$, then

$$\begin{aligned}
\langle \psi, \theta \rangle_q &= \lim_{k \to \infty} \langle (a_0^*)^n \varphi_k, \theta \rangle_q = \lim_{k \to \infty} \langle \varphi_k, \xi \rangle_q \\
&= \frac{1}{[n]_q!} \lim_{k \to \infty} \langle \varphi_k, P_n(a_0^* a_0)\xi \rangle_q \\
&= \frac{1}{[n]_q!} \lim_{k \to \infty} \langle (a_0^*)^n \varphi_k, (a_0^*)^n \xi \rangle_q \\
&= \frac{1}{[n]_q!} \langle \psi, (a_0^*)^n \xi \rangle_q = 0
\end{aligned}$$

because $(a_0^*)^n \xi \in \mathcal{K}_n \perp \psi$. The claim, (4), follows by induction.

We define an operator $V \colon \mathcal{F}_q(\mathbb{C}) \otimes \mathcal{K} \to \mathcal{F}_q(\mathbb{C}^2)$ by:

$$V(e_n \otimes \varphi) := (a_0^*)^n \varphi.$$

The operator V is an isomorphism since its range is dense by the above, and, for all $\varphi, \xi \in \mathcal{K}$,

$$\begin{aligned}
\langle V(e_n \otimes \varphi), V(e_m \otimes \xi) \rangle_q &= \langle (a_0^*)^n \varphi, (a_0^*)^m \xi \rangle_q \\
&= \delta_{n,m} \langle \varphi, P_n(a_0^* a_0)\xi \rangle_q \\
&= \delta_{n,m} [n]_q! \langle \varphi, \xi \rangle_q \\
&= \langle e_n \otimes \varphi, e_m \otimes \xi \rangle_q.
\end{aligned}$$

$\square$

Lemma 1. The operator W has the following properties:
   (i) W is unitary and self adjoint,
   (ii) $\gamma(X_0) = W X_0 W$,
   (iii) $W\varphi = \varphi$ for all $\varphi \in \mathcal{K}$, in particular $W\Omega = \Omega$,
   (iv) $W(X_0 \varphi) = \sum_{k=1}^{\infty} w_{k1} (a_0^*)^k \varphi$ for all $\varphi \in \mathcal{K}$.

Proof. Property (i) is clear from the definition of W since $\tilde{W}$ is unitary and self-adjoint.

To prove property (ii), note that, for $\varphi \in \mathcal{K}$ and $n \in \mathbb{N}$, $V(a^* \otimes \mathbb{1})(e_n \otimes \varphi) = V(e_{n+1} \otimes \varphi) = a_0^* V(e_n \otimes \varphi)$, so $V(a^* \otimes \mathbb{1})V^* = a_0^*$ and $V(X \otimes \mathbb{1})V^* = X_0$. It follows that

$$
\begin{aligned}
W X_0 W &= W V(X \otimes \mathbb{1})V^* W \\
&= V(\tilde{W} \otimes \mathbb{1})(X \otimes \mathbb{1})(\tilde{W} \otimes \mathbb{1})V^* \\
&= V(\gamma(X) \otimes \mathbb{1})V^* \\
&= V\gamma(X \otimes \mathbb{1})V^* = \gamma(X_0).
\end{aligned}
$$

Property (iii) is immediate from definitions:

$$
W\varphi = V(\tilde{W} \otimes \mathbb{1})(e_0 \otimes \varphi) = V(e_0 \otimes \varphi) = \varphi
$$

for all $\varphi \in \mathcal{K}$.

The proof of property (iv) is also immediate from definitions:

$$
W(X_0\varphi) = W(a_0^*\varphi) = V(\tilde{W} \otimes \mathbb{1})(e_1 \otimes \varphi) = V(\tilde{W}e_1 \otimes \varphi)
$$

$$
= \sum_{k=1}^{\infty} w_{k1} V(e_k \otimes \varphi) = \sum_{k=1}^{\infty} w_{k1}(a_0^*)^k \varphi.
$$

$\square$

We now turn to the proof of theorem 1.

Proof. First we calculate the fourth moment of $X_0 + X_1$. Since $X(f_0 + f_1)$ is q-Gaussian with variance 2 we have

$$
\begin{aligned}
\langle \Omega, (X_0 + X_1)^4 \Omega \rangle_q &= (\sqrt{2})^4 \langle \Omega, X_0^4 \Omega \rangle_q = 4\|X_0^2\Omega\|_q^2 \\
&= 4(\|\Omega\|_q^2 + \|f^{\otimes 2}\|_q^2) = 4(1 + [2]_q) \\
&= 8 + 4q
\end{aligned}
$$

a linear interpolation between 8 and 12 for q varying between 0 and 1. We now turn to the calculation of the fourth moment of $\gamma(X_0) + X_1$:

$$
\langle \Omega, (\gamma(X_0) + X_1)^4 \Omega \rangle_q = \|(\gamma(X_0) + X_1)^2\Omega\|_q^2.
$$

For this we need the following:

$$
\begin{aligned}
\gamma(X_0)^2\Omega &= W X_0^2 W\Omega = \Omega + W f_0^{\otimes 2} \\
X_1^2\Omega &= \Omega + f_1^{\otimes 2} \\
\gamma(X_0)X_1\Omega &= W X_0 W X_1\Omega = W X_0 X_1\Omega \\
&= \sum_{k=1}^{\infty} w_{k1}(a_0^*)^k X_1\Omega = \sum_{k=1}^{\infty} w_{k1} f_0^{\otimes k} \otimes f_1 \\
X_1\gamma(X_0)\Omega &= X_1 W X_0\Omega = \sum_{k=1}^{\infty} w_{k1} f_1 \otimes f_0^{\otimes k}
\end{aligned}
$$

from which it is easy to deduce that

$$
\|(\gamma(X_0) + X_1)^2\Omega\|_q^2 = \|(\gamma(X_0)^2 + X_1^2)\Omega\|_q^2 + \|(\gamma(X_0)X_1 + X_1\gamma(X_0))\Omega\|_q^2. \tag{5}
$$

The first term on the right-hand side of (5) is found to be:

$$
\|(\gamma(X_0)^2 + X_1^2)\Omega\|_q^2 = 4\|\Omega\|_q^2 + \|f_0^{\otimes 2}\|_q^2 + \|f_1^{\otimes 2}\|_q^2 = 4 + 2[2]_q = 6 + 2q.
$$

The second term on the right-hand side of (5) yields:

$$\|(\gamma(X_0)X_1 + X_1\gamma(X_0))\Omega\|_q^2 = \sum_{k=1}^{\infty} w_{k1}^2 \|(f_0^{\otimes k} \otimes f_1 + f_1 \otimes f_0^{\otimes k})\|_q^2$$

$$= \sum_{k=1}^{\infty} w_{k1}^2 (2\|f_0^{\otimes k} \otimes f_1\|_q^2 + 2\langle f_0^{\otimes k} \otimes f_1, f_1 \otimes f_0^{\otimes k}\rangle_q)$$

$$= 2\sum_{k=1}^{\infty} w_{k1}^2 (1 + q^k)[k]_q!$$

$$= 2 + 2\sum_{k=1}^{\infty} w_{k1}^2 q^k [k]_q!.$$

To prove the theorem it remains to show that

$$\sum_{k=1}^{\infty} w_{k1}^2 q^k [k]_q! < q \qquad \text{for } q \in (0, 1).$$

To this end, note that $q^k < q$ for $k \geqslant 2$ and $q \in (0, 1)$, so

$$\sum_{k=1}^{\infty} w_{k1}^2 (q^k - q)[k]_q! < 0$$

from which it follows that

$$\sum_{k=1}^{\infty} w_{k1}^2 q^k [k]_q! < q \sum_{k=1}^{\infty} w_{k1}^2 [k]_q! = q\|\tilde{W}e_1\|_q^2 = q.$$

We conclude that $\langle \Omega, (\gamma(X_0) + X_1)^4 \Omega \rangle_q < \langle \Omega, (X_0 + X_1)^4 \Omega \rangle_q$ for $q \in (0, 1)$. $\qquad \square$

The content of theorem 1 is shown graphically in figure 2 where the fourth moment of $X_0 + X_1$ and a numerical approximation of the fourth moment of $\gamma(X_0) + X_1$ are plotted.
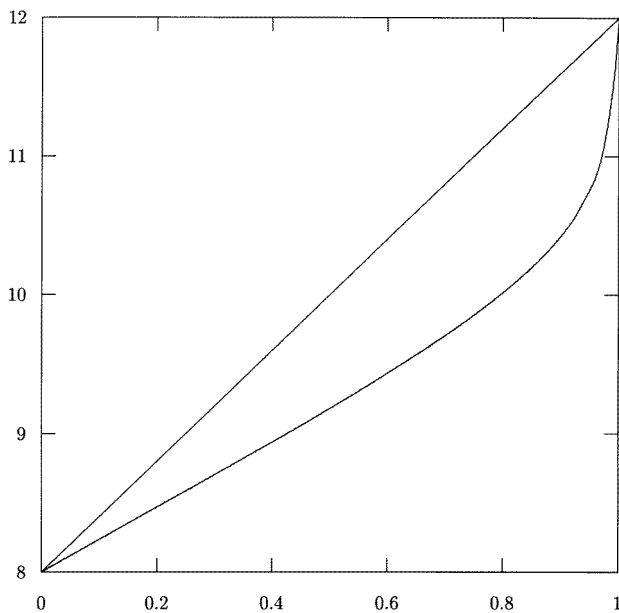

Figure 2. The fourth moment of $X_0 + X_1$ and $\gamma(X_0) + X_1$ for $q \in (0, 1)$.

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